

# An Information Theoretic Approach to Reflectional Symmetry Detection

Agata Migalska  
Wrocław University of Technology  
Wrocław, Poland  
Email: agata.migalska@pwr.edu.pl

John Lewis  
Victoria University of Wellington  
Wellington, New Zealand  
Email: john.lewis@vuw.ac.nz

**Abstract**—Symmetry is an omnipresent transformation in both nature and man-made objects. It is remarkable how human beings are capable of detecting and recognizing symmetries in the surrounding world without hesitation and without much apparent mental effort. On the other hand, teaching a machine to perform the same task has been challenging, resulting in a variety of approaches and algorithms. In this paper we appeal to information theory to obtain a novel and general principle for symmetry detection. Folding an image in half along a line that coincides with an axis of reflectional symmetry preserves the statistics of the image, whereas folding along any other line alters these statistics to become more Gaussian. Symmetric transforms can thus be detected as those that have the largest negentropy. Experimental evaluation shows that our algorithm properly detects the symmetry axes within synthetic and natural images and is applicable for reflectional symmetry of an arbitrary order.

**Index Terms**—symmetry detection, negentropy, angle of reflection

## I. INTRODUCTION

Symmetry can be defined as a geometric transformation, other than an identity, which, applied to a given mathematical object, returns an object identical to the input [1].

Seldom does symmetry occur by accident. If two regions are symmetric it is highly probable that these regions are related in the real world, or even that they belong to the same object. Therefore, by detecting symmetry it is possible to start grouping or segmenting the image without prior knowledge of image content. As many objects exhibit some degree of symmetry, symmetry detection can be to the benefit of the computer vision systems by providing information significant for image analysis and understanding, shape classification or recognition. An additional benefit is derived from symmetry detection in the field of image compression. The existence of self symmetry of the image lays the foundation for the fractal image compression, while bilateral symmetry has been exploited in medical image lossless compression [2].

Perception of symmetry in humans and other animals has been well-studied and the concluding remarks are that symmetry is both attractive and biologically significant [3], [4]. Moreover, it can be detected by simple visual processes [3]. On the other hand, despite the significant research effort made there is still a need for a robust, widely applicable symmetry detector [5].

## A. Related Work

Symmetry has been thoroughly studied in literature, starting from philosophical through mathematical to computational points of view. A theoretical analysis of symmetry, including the two former outlooks, can be found in a celebrated book by Weyl [1]. We shall, however, focus our attention on the computational approaches to symmetry detection and recognition, and on methods for detection of the reflectional symmetry axes in particular.

Existing methods for symmetry detection can be grouped into several categories. One approach rests on the characteristics of the Fourier transform, that the symmetry of images in their frequency domain representations is preserved.

Sun et al. [6] proposed a reflectional symmetry detection algorithm that employs the gradient orientation histogram of the input gray-scale image to obtain the directions of the symmetry axes. Fast Fourier transform is used to obtain the convolution function from the gradient orientation histogram of the input image. The peaks in the obtained convolution function correspond to the orientations of symmetry axes.

A similar method, in terms of use of autocorrelation as a tool for reflective symmetry detection, was presented in the work of Kazhdan et al. [7]. A reflective symmetry descriptor is introduced to assess the level of symmetry, and is obtained by decomposing the image into concentric circles and computing the reflective symmetry descriptors on each of the circles.

Another approach was proposed by Derrode and Ghorbel [8], who applied the Analytical Fourier-Mellin transform (AFMT) in order to detect and localize the symmetry axes in gray-level images. In their method, a functional is constructed from the Euclidean distance between the AFMT representation of two objects. The location of its minimum determines the scale and rotation that should be applied on the first object to match the second.

Lucchese [9] presented an algorithm that operates on the image representation in Cartesian coordinates and does not necessitate computing its polar discrete Fourier transform. The task of classifying reflectional and rotational symmetries is accomplished by looking at the point-wise zero crossings of the difference of the Fourier transform magnitudes along rays. However, whether the method could be applied to natural or noisy images was not evaluated.

Keller and Shkolnisky [10] proposed using a pseudo-polar grid for Fourier transform computation. Their algorithm for rotational and reflectional symmetries detection is based on the properties of an angular correlation (AC), which is shown to be a periodic signal having frequency related to the order of symmetry. Retrieval of the tilt angle of reflection axis is made by registering an image to its upside down version.

A different class of intensity-based algorithms utilizes the numerical moments calculated from the image.

Shen et al. [11] presented a Shen-Ip Symmetry Detector for detecting both reflections and rotations, that is based on first three nonzero generalized complex (GC) moments. In their method, the fold number is determined by calculating the biggest common factor of all orders making GC moments nonzero, and the fold axes are defined by the phase and the order of the combined moment which is generated from all the nonzero GC moments.

An approach for testing image symmetries utilizing the Zernike moments was proposed by Bissantz et al.[12]. The proposed test statistics are constructed by expressing the symmetry conditions in terms of restrictions on the Zernike moments. This work was extended by Pawlak [13], who proposed an estimator for the tilt angle of reflection. It is shown that an angle, for which the distance between the observed function and its symmetric counterpart attains the minimum, is the true angle of reflection axis. However, the main drawback of the methods based on moments is their high computational complexity.

Almost all of the aforementioned methods suffer from the necessity of transforming the image from Cartesian into the polar coordinate system. While discretisation of the continuous image function is already a source of error, the transformation from one coordinate system to another introduce an additional geometric error. In order to alleviate this problem a pseudo-polar grid of concentric squares was employed in [10], while in [12] a polar pixel tiling scheme [14] is suggested to improve the accuracy of the Zernike moments computation. However, whilst resolving accuracy issues, these methods inherently reduce the clarity of the proposed methods and increase their computational complexity.

### B. Contribution

The algorithm we present identifies the global reflectional symmetries of gray-scale images. An input image is folded along a hypothesized symmetry axis passing through the centre of the image. The intensities within the folded image are the averaged intensities of the overlapping pixels from both halves of the input image. Supposing that the folded image is a realization of a random process the non-gaussianity of its probability distribution is observed. Our method is based on an observation that if the hypothesized tilt angle of reflection axis is correct and both halves of the input image are derived from one probability distribution then the probability distribution of the folded image preserves the statistics of the image. However, if the hypothesized angle is incorrect, the probability distribution of the folded image becomes more Gaussian.

Since it is commonly believed that images are very non-Gaussian, high dimensional, continuous signals [15], the most non-Gaussian distribution corresponds to the true tilt angle of the symmetry axis. This distance to non-normality is measured by means of negentropy, a measure previously employed in Independent Component Analysis (ICA)[16], [17] for separating a multivariate signal into additive subcomponents.

Our contribution in this paper is *NegReflection* - an accurate and robust method for reflectional symmetry detection. Unlike other existing methods, the algorithm operates on the Cartesian grid and does not require any conversion to polar image representation. Moreover, as shall be presented in the experimental evaluation section of the paper, our method can be successfully applied to detect symmetries of an arbitrary order, i.e. having one or more axes of symmetry, as well as to detect almost-symmetries in natural images. The novelty of the proposed method lays in measuring the amount of non-gaussianity within the folded images to retrieve an information on symmetry. Such a concept, to our best knowledge, has not been utilized before. Given the appealing properties of the discovered relationship it is hoped that its further investigation will lead to development of robust, widely applicable symmetry detector.

The rest of the paper is organized as follows. Section II introduces the proposed method complemented by the necessary theoretical background. Section III outlines the algorithm and implementation details. Finally, sections IV and V present experimental results and concluding remarks, respectively.

## II. THEORETICAL PRELIMINARIES

Let us consider a two-dimensional intensity image  $I$  recorded on a square, equally-spaced Cartesian grid of size  $n \times n$ . Each element of the grid, a pixel, is a discrete sample from a continuous image function whose value is in  $[0, 1]$ .

### A. Image Symmetry

A two-dimensional function is said to be reflectionally symmetric if it is invariant to reflection with respect to one or more straight lines, the reflection axes. A formal definition of reflectional symmetry is given below.

*Definition 1 (Reflectional symmetry):* A function  $\psi : \mathbf{R}^2 \rightarrow \mathbf{R}$  is reflectionally symmetric with respect to the vector  $(\cos \theta_0, \sin \theta_0)$  if

$$\psi(x, y) = \psi(S_{\theta_0}(x, y)) \quad (1)$$

where

$$S_{\theta_0}(x, y) = \begin{pmatrix} \cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & -\cos 2\theta_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2)$$

$\theta_0$  is the tilt angle of the reflection axis of  $\psi$ . A function  $\psi$  has reflectional symmetry of order  $N$  if there are  $N$  angles  $\theta_n$  that satisfy Equation 1.

A graphical illustration of reflectional symmetry is presented in Figure 1a, where the symmetry axis passes through the origin of the coordinate system and is rotated counter-clockwise about the origin by the tilt angle  $\theta_0$ . In our scheme,

we make an assumption that the axis of reflection passes through the center of an image and is rotated about the center of an image by  $\theta_0$ , as given in Figure 1b. An axis of reflection is given by  $y = ax + b$  where  $\theta_0$  is a tilt angle, and  $a = \tan \theta_0$  and  $b = \frac{n}{2}(1 - a)$  are the slope and intercept, respectively.

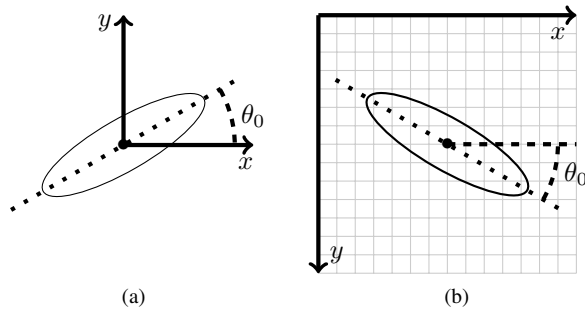


Fig. 1. 1a) Symmetry axis passing through the origin of the Cartesian coordinate system and rotated counter-clockwise by tilt angle  $\theta_0$ . 1b) Image placement with respect to the origin of the coordinate system with row zero at the top. Symmetry axis, rotated clockwise by tilt angle  $\theta_0$ , passes through the centre of an image.

A line passing through the centre of an image  $I$  splits this image into two halves, the one above the line represented by  $I_U$  and the other below the line represented by  $I_L$ . Let us now imagine that we are taking an image and folding it along this line, which results in an image half the size of the original one. For each pixel in the resulting image its intensity is an average of an original pixel at a given position and of a folded one, i.e. reflected by the above-mentioned line - an axis of reflection, and is given by

$$I_S(i, j) = \frac{1}{2} (I_U(i, j) + I_L(S_{\theta_0}(i, j))), \quad (3)$$

where  $S_{\theta_0}$  is a symmetry transformation defined in Equation 2 and  $i, j \in \{1, \dots, n\}$ .

In Figure 2 an example of the above-mentioned folding is given. The upper half of an image and the reflected lower half are shown in Figure 2a and 2b, respectively, while the result of taking the average of both is presented in Figure 2c.

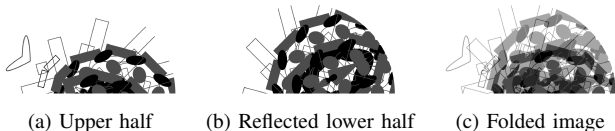


Fig. 2. Two halves of the image (2a and 2b) and their average (2c).

### B. Distribution of the Folded Image

Suppose the half-images  $I_U$  and  $I_L$  are the  $\frac{n^2}{2}$ -element samples from some one-dimensional probability distributions. What is the distribution of  $I_S$ ? If the image  $I$  is perfectly reflectionally symmetric and folded along the symmetry axis, then  $I_S(i, j) = I_U(i, j) = I_L(S_{\theta_0}(i, j))$ . While from Central Limit Theorem it arises that an addition of a large number of independent random variables is approximately normal [18],

it is not reasonable to consider  $I_L$  and  $I_U$  independent in this case. On the other hand, an incorrectly chosen transform  $S_{\theta}$  will result in the addition of regions that are not symmetrically related and can be considered approximately independent. Therefore, the probability distribution of the folded image becomes more Gaussian. Further, as outlined in the Appendix, the distribution will tend to change gradually as the identified transform  $S_{\theta_0}$  changes from the true symmetry to a transform that puts unrelated regions in correspondence.

Thus, we hypothesize that in order to determine the true tilt angle of symmetry axis an amount of gaussianity in the distribution of the image  $I_S$ , given a hypothesized tilt angle  $\theta$ , is to be observed. To assess this amount of gaussianity we shall turn our attention to entropy, a basic concept in information theory, and negentropy - its counterpart that gained popularity in Independent Component Analysis (ICA)[16], [17] as a measure of non-gaussianity.

Differential entropy, an extension of Shannon entropy to continuous probability distributions, is a measure of amount of the information associated with a continuous random variable. The more “random”, i.e. unpredictable and unstructured, the variable is, the larger its entropy. The differential entropy of a random variable  $\mathbf{y}$  with density  $p_{\mathbf{y}}(\eta)$  is defined as

$$H(\mathbf{y}) = - \int p_{\mathbf{y}}(\eta) \log p_{\mathbf{y}}(\eta) d\eta. \quad (4)$$

A fundamental result of information theory is that a Gaussian variable has the largest differential entropy among all random variables under the constraints of mean and variance, meaning that the Gaussian distribution is the most random or the least structured of all continuous distributions (whereas the highest entropy discrete distribution is the uniform distribution). In order to determine how close the distribution of  $\mathbf{y}$  is to the normal distribution negentropy could be used. The negentropy of a random variable  $\mathbf{y}$  is defined as

$$J(\mathbf{y}) = H(\mathbf{y}_{gauss}) - H(\mathbf{y}) \quad (5)$$

where  $\mathbf{y}_{gauss}$  is a Gaussian random variable of the same mean and variance as  $\mathbf{y}$ . Negentropy can be thought of as a measure of non-gaussianity or degree of structuredness of the distribution of  $\mathbf{y}$ . Negentropy is non-negative, reaching zero if and only if  $\mathbf{y}$  has a normal distribution, and is invariant under invertible linear transformations.

Estimating negentropy from the definition is computationally difficult. Fortunately, to our aid comes the approximation of negentropy by nonpolynomial functions [17, pg. 118]:

$$J(y) \approx k_1 (E \{y \exp(-y^2/2)\})^2 + k_2 \left( E \{ \exp(-y^2/2) \} - \sqrt{\frac{1}{2}} \right)^2 \quad (6)$$

where  $k_1 = 36 / (8\sqrt{3} - 9)$  and  $k_2 = 24 / (16\sqrt{3} - 27)$  [17, pg. 119]. It should be noted that in [17] two approximation formulas are given. The above formula was chosen due to the smaller approximation error (see [17, pg.120-121]) as well as

due to higher smoothness of the obtained negentropy function in the course of experimental verification.

Equipped with the above tools we may now state the primary result presented in this paper.

*Proposition 1:* Let  $I_S$  be a folded image, as defined in Equation 3, of a reflectionally symmetric image  $I$ . The tilt angle of the reflection axis, denoted as  $\theta_0$ , can be estimated as

$$\hat{\theta}_0 = \arg \max_{\theta \in \Theta} J(I_S(\theta)), \quad (7)$$

where  $\Theta$  is the set of hypothesized tilt angles.

Thus, finding the tilt angle of reflection axis is equivalent to finding the angle for which the negentropy of the folded image is maximized.

### III. IMPLEMENTATION DETAILS

To demonstrate Propostion 1, a set of integer valued angles  $\Theta = \{0, 1, 2, \dots, 179\}$  in degrees is considered. It should be noted that due to periodicity of the tangent function,  $\tan \alpha = \tan(\alpha + k\pi)$ ,  $k \in \mathbb{Z}$ ,  $\alpha \in [0, \pi)$ , the set  $\Theta$  is limited to two quarters of the circle.

The *NegReflection* algorithm is as follows.

For every hypothesized angle  $\theta$  within the set  $\Theta$ :

- 1) a reflection axis is calculated and  $I_U$  and  $S_\theta(I_L)$  are determined,
- 2)  $I_S$  is obtained by averaging values in  $I_U$  and  $S_\theta(I_L)$ ,
- 3)  $I_S$  is normalized to zero-mean and unit variance,
- 4) the negentropy of  $I_S$  is approximated by formula given in Equation 6.

Finally, following the result presented in Proposition 1, the angle that maximizes the negentropy is selected as an estimated tilt angle of reflection axis.

The pseudocode of the *NegReflection* algorithm is given in Algorithm 1.

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#### Algorithm 1 NegReflection Algorithm

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1: procedure NEGREFLECTION( $I, \Theta$ )
2:    $\theta_0 \leftarrow \infty$ 
3:    $J_0 \leftarrow \infty$ 
4:   for all  $\theta \in \Theta$  do
5:      $I_U, S_\theta(I_L) \leftarrow \text{REFLECT}(I, \theta)$ 
6:      $I_S \leftarrow (I_U + S_\theta(I_L)) / 2$ 
7:      $I_S \leftarrow \text{NORMALIZE}(I_S)$ 
8:      $J \leftarrow \text{NEGENTROPY}(I_S)$  ▷ Equation 6
9:     if  $J < J_0$  then
10:        $\theta_0 \leftarrow \theta$ 
11:        $J_0 \leftarrow J$ 
12:     end if
13:   end for
14:   return  $\theta_0$  ▷ Tilt angle
15: end procedure

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The only transformation of an input image is performed during the `reflect` operation, when the upper and the reflected lower matrices are being constructed. Given an image

matrix  $I$  and a tilt angle  $\theta$ , the slope  $a$  and intercept  $b$  of a reflection axis are calculated as well as the slope  $a_\perp$  of a line perpendicular to the reflection axis. For each pixel, the intercept  $b_\perp$  of a perpendicular line is determined as well as the coordinates  $(x', y')$  of the point that should be reflected onto the currently processed pixel location  $(i, j)$ . If the point to be reflected is located above the symmetry axis it is stored at  $(i, j)$  location of both resulting matrices. If it is located on the symmetry axis it is stored only in  $I_U$ .

### IV. EXPERIMENTAL RESULTS

Experimental evaluation of the proposed method was performed on synthetic and natural images. For each test image the highest negentropy values and their corresponding angles were determined. Resolved symmetry axes were marked on each test image for visual verification. The obtained results are discussed below.

#### A. Bilateral Symmetry Detection

Bilateral symmetry, i.e. of degree 1, was tested on the set of synthetic images having symmetry axes rotated by various angles. Examples of obtained results are given in Figure 3. Tilt angle is properly recognized to be  $60^\circ$  for Image 1 (3a) and  $160^\circ$  for Image 2 (3b). Plots of negentropy as a function of angle for both images are shown in Figures 3c and 3d, respectively.

#### B. Higher Order Symmetry Detection

Higher orders of reflectional symmetry were also tested on the set of synthetic images, with the order of symmetry ranging from 2 to 9. The results, presented in Figure 4, show that negentropy in these cases resembles a periodic function with the highest peaks corresponding to the angles of symmetry axes.

#### C. Symmetries in Natural Images

Finally, in order to evaluate performance of our method on images that are not perfectly symmetric, the method was tested on a set of natural images. Obtained results are given in Figure 5. Even though the plots of negentropy are more noisy, compared to the results for synthetic images, maximal values are still observed for the approximately correct angles.

### V. CONCLUSION

We presented a 2D symmetry detection algorithm which detects the reflection axes of symmetry within gray-scale images. An estimate of the tilt angle of the reflection axis is proposed as such that maximizes the negentropy of the image folded along this axis. This formulation is based on the observation that if an image is split by a true reflection axis so that both its halves are drawn from the same, non-Gaussian distribution then the distribution of the folded image is unchanged and also non-Gaussian. However, if a selection of a symmetry axis is incorrect and two halves of an input image are not symmetric then the distribution of the folded image becomes more Gaussian. The approach is shown to be accurate and effective in recovering centered reflectional symmetries

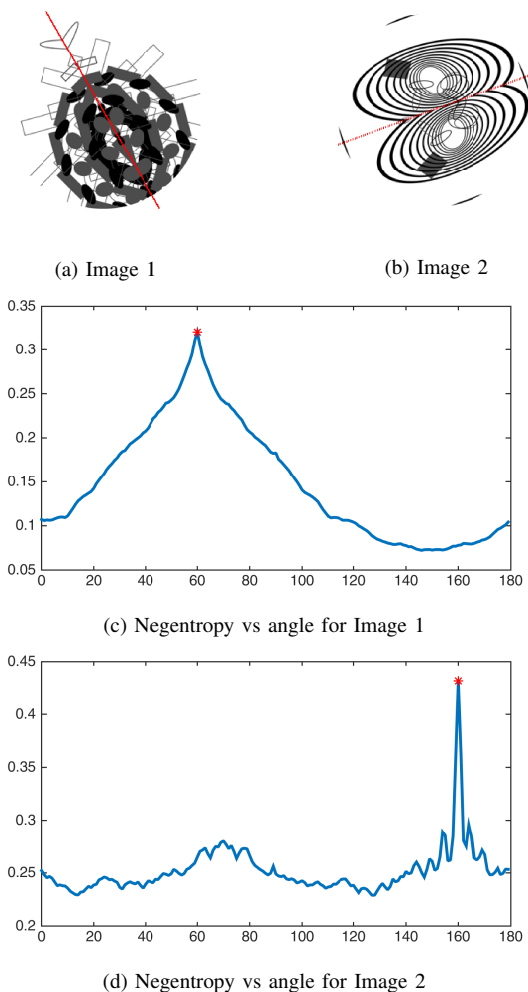


Fig. 3. Results obtained for images with bilateral symmetry. Maximum negentropy is attained at the tile angle for the symmetry axis. Detected axes are shown in 3a and 3b.

of arbitrary order in gray-scale images, both synthetic and natural. The formulation of symmetry detection in terms of information theory also may have interesting consequences that are not explored in this paper. For example, the probability density function (pdf) of an image region is insensitive to permuting the pixel locations, and thus our formulation might handle symmetries that are only statistical in character – unlike measures such as simply minimizing a difference between corresponding pixels.

The work reported here is only targeted at validating the proposed principle for symmetry identification, and we did not consider important considerations such as optimization strategies or finding the centre of symmetry. It is hoped that further algorithmic development and analysis of the properties of the distributions underlying symmetric regions within images will eventually lead to development of a robust, widely applicable symmetry detector.

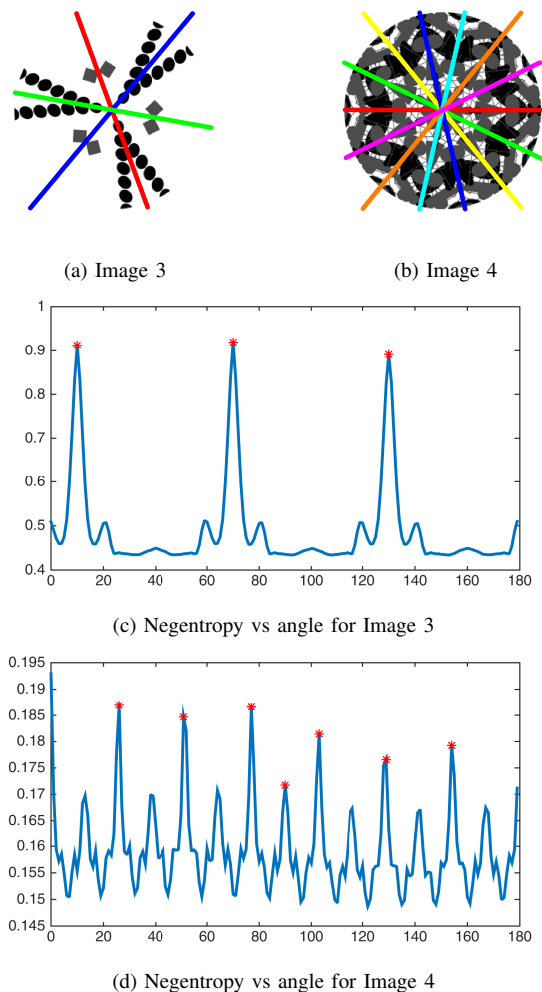


Fig. 4. Results obtained for images with higher orders of symmetry. The observed peaks of negentropy correspond to the tilt angles of the symmetry axes. Detected axes are shown in 4a and 4b.

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#### APPENDIX

Denote the discrete pdf (histogram) of a greyscale image as

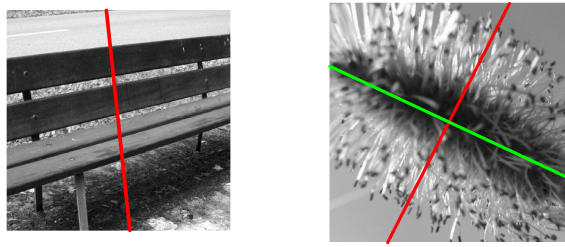
$$P_{I_S}(v) = \frac{1}{N} \sum_{x,y \in \Omega} \delta(I(x,y), v)$$

where  $\Omega$  is the support of the image  $I$ ,  $N$  is the number of pixels,  $v$  is the grey value, and  $\delta(a,b)$  is the Kronecker delta. The pdf of  $I_S$  in (3) is then

$$P_{I_S}(v) = \frac{1}{N} \sum_{x,y \in \Omega} \delta\left(\frac{1}{2}I_U(x,y) + \frac{1}{2}I_L(S_{\theta_0}(x,y)), v\right)$$

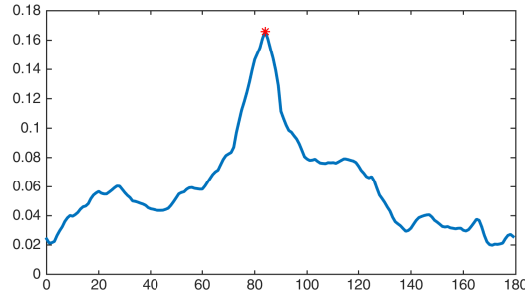
Expanding  $I_L$  in a Taylor series up to the linear term and denoting  $\mathbf{p} \equiv (x,y)$ ,

$$I_L(S_{\theta}(\mathbf{p})) \approx I_L(S_{\theta_0}(\mathbf{p})) + \nabla I_L(S_{\theta_0}(\mathbf{p})) J_S(\mathbf{p}, \theta_0) \delta\theta$$

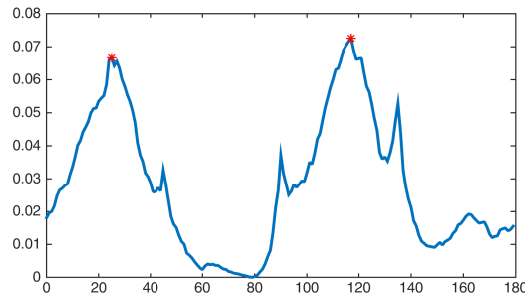


(a) Image 5

(b) Image 6



(c) Negentropy vs angle for Image 5



(d) Negentropy vs angle for Image 6

Fig. 5. Results obtained for natural images with reflectional symmetry. The observed peaks of negentropy correspond to the tilt angles of the symmetry axes. Detected axes are shown in 5a and 5b.

where  $\delta\theta = \theta - \theta_0$  and  $J_S(\mathbf{p}, \theta_0)$  is the Jacobian of the symmetry transform with respect to  $\theta$ . If the expansion is centered about the correct symmetry  $S_{\theta_0}$  then  $I_U(x, y) = I_L(S_{\theta_0}(x, y))$ . Substituting this into the discrete pdf expression,

$$P_{I_S}(v) = \frac{1}{N} \sum_{x, y \in \Omega} \delta \left( I_U(x, y) + \frac{1}{2} \nabla I_L(S_{\theta_0}(\mathbf{p})) J_S(\mathbf{p}, \theta_0) \delta\theta, v \right) \quad (8)$$

It can be seen that a small change to the symmetry parameter  $\theta$  leads to a small change in the histogram  $P_I(v)$  provided that the image is smooth ( $\nabla I_L$  is small). Further, entropy is a continuous function of the pdf, so we expect that our information theoretic symmetry measure decays gradually from a peak at the correct symmetry, as seen in the figures.

It is often assumed that images are piecewise smooth [19], so the gradient  $\nabla I$  will be small at most pixels in the sum in equation (8). This property is not required to identify the

correct symmetry, but if present can allow the application of optimization algorithms that assume some regularity of the error function.

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